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Petrov classification of the cylindrically symmetric gravitational field

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Abstract. We consider in this paper the most general cylindrically symmetric space–time from the point of view of Petrov classification. The Pirani matrix has been classified into six distinct cases. The Segre characteristics reveal that the cylindrically symmetric metric belongs to either Petrov type I or Petrov type II. It is also found that when the metric potentials are functions of time alone the space–time admits perfect fluid distribution.

1. Introduction

The most general cylindrically symmetric space–time is given by (Marder 1958)

$$ds^2 = A^2(dt^2 - dx^2) - B^2\,dy^2 - C^2\,dz^2 \tag{1}$$

where A, B and C are functions of x and t only. The surviving components of the mixed Ricci tensor are as follows:

$$\begin{aligned}
R_1{}^1 &= \frac{1}{A^2}\left\{\frac{A_{44}-A_{11}}{A} - \frac{B_{11}}{B} - \frac{C_{11}}{C} + \frac{1}{A}\left(\frac{A_1B_1+A_4B_4}{B} + \frac{A_1C_1+A_4C_4}{C} + \frac{A_1{}^2-A_4{}^2}{A}\right)\right\}\\
R_2{}^2 &= \frac{1}{A^2}\left(\frac{B_{44}-B_{11}}{B} + \frac{B_4C_4-B_1C_1}{BC}\right)\\
R_3{}^3 &= \frac{1}{A^2}\left(\frac{C_{44}-C_{11}}{C} + \frac{B_4C_4-B_1C_1}{BC}\right)\\
R_4{}^4 &= \frac{1}{A^2}\left\{\frac{A_{44}-A_{11}}{A} + \frac{B_{44}}{B} + \frac{C_{44}}{C} - \frac{1}{A}\left(\frac{A_1C_1+A_4C_4}{C} + \frac{A_1B_1+A_4B_4}{B} + \frac{A_4{}^2-A_1{}^2}{A}\right)\right\}\\
R_1{}^4 &= -R_4{}^1 = \frac{1}{A^2}\left(\frac{B_{14}}{B} + \frac{C_{14}}{C}\right) - \frac{1}{A^3}\left(\frac{A_1B_4+A_4B_1}{B} + \frac{A_1C_4+A_4C_1}{C}\right).
\end{aligned} \tag{2}$$

The components of the Weyl conformal curvature tensor C_{hijk} for the metric (1) are as follows:

$$\begin{aligned}
C_{14}{}^{14} = C_{23}{}^{23} &= \frac{1}{6A^2}\left(\frac{B_{44}-B_{11}}{B} + \frac{C_{44}-C_{11}}{C} - 2\frac{A_{44}-A_{11}}{A}\right.\\
&\qquad\left. + 2\frac{A_4{}^2-A_1{}^2}{A^2} + 2\frac{B_1C_1-B_4C_4}{BC}\right)\\
C_{12}{}^{12} = C_{34}{}^{34} &= \frac{1}{6A^2}\left(\frac{A_{44}-A_{11}}{A} + \frac{2B_{11}+B_{44}}{B} - \frac{2C_{44}+C_{11}}{C} + 3\frac{A_1C_1+A_4C_4}{AC}\right.\\
&\qquad\left. + \frac{A_1{}^2-A_4{}^2}{A^2} - 3\frac{A_1B_1+A_4B_4}{AB} + \frac{B_4C_4-B_1C_1}{BC}\right)\\
C_{13}{}^{13} = C_{24}{}^{24} &= \frac{1}{6A^2}\left(\frac{A_{44}-A_{11}}{A} + \frac{2C_{11}+C_{44}}{C} - \frac{2B_{44}+B_{11}}{B} + 3\frac{A_1B_1+A_4B_4}{AB}\right.\\
&\qquad\left. + \frac{A_1{}^2-A_4{}^2}{A^2} - 3\frac{A_1C_1+A_4C_4}{AC} + \frac{B_4C_4-B_1C_1}{BC}\right)
\end{aligned} \tag{3}$$

$$C_{34}{}^{13} = -C_{24}{}^{12} = \frac{1}{2A^2}\left\{\frac{B_{14}}{B} - \frac{C_{14}}{C} + \frac{1}{A}\left(\frac{C_1A_4+A_1C_4}{C} - \frac{A_1B_4+B_1A_4}{B}\right)\right\}.$$

2. Petrov–Pirani classification

Let $\lambda_{(a)}{}^h$ be a set of four mutually orthogonal unit vectors associated with an event in space–time. From C_{hijk} we can construct a scalar invariant with the help of $\lambda_{(a)}{}^h$ as follows:

$$C_{(abcd)} = C_{hijk}\lambda_{(a)}{}^h\lambda_{(b)}{}^i\lambda_{(c)}{}^j\lambda_{(d)}{}^k. \tag{4}$$

$C_{(abcd)}$ are called physical components of the conformal curvature tensor. We choose the tetrad $\lambda_{(a)}{}^h$ as

$$\lambda_{(a)}{}^h = \text{diagonal}\left(-\frac{1}{A}, -\frac{1}{B}, -\frac{1}{C}, \frac{1}{A}\right). \tag{5}$$

With this tetrad the non-vanishing physical components of C_{hijk} for the metric (1) are

$$\begin{aligned} C_{(1212)} &= C_{12}{}^{12}, & C_{(2424)} &= -C_{24}{}^{24} \\ C_{(1313)} &= C_{13}{}^{13}, & C_{(3434)} &= -C_{34}{}^{34} \\ C_{(2323)} &= C_{23}{}^{23}, & C_{(1414)} &= -C_{14}{}^{14} \\ C_{(1334)} &= -C_{34}{}^{13}, & C_{(1224)} &= -C_{24}{}^{12}. \end{aligned} \tag{6}$$

If we relabel the index pairs (ab), (cd) according to the scheme

$$\begin{array}{lcccccc} (ab): & 23 & 31 & 12 & 14 & 24 & 34 \\ A\;: & 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

the λ matrix can be written as

$$C_{[AB]} - \lambda\eta_{[AB]} = \begin{bmatrix} \alpha-\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta-\lambda & 0 & 0 & 0 & b \\ 0 & 0 & \gamma-\lambda & 0 & b & 0 \\ 0 & 0 & 0 & -(\alpha-\lambda) & 0 & 0 \\ 0 & 0 & b & 0 & -(\beta-\lambda) & 0 \\ 0 & b & 0 & 0 & 0 & -(\gamma-\lambda) \end{bmatrix} \tag{7}$$

where $\eta_{AB} = \text{diagonal}\,(1, 1, 1, -1, -1, -1)$,

$$\alpha = C_{23}{}^{23}, \qquad \beta = C_{13}{}^{13}, \qquad \gamma = C_{12}{}^{12}, \qquad b = -C_{24}{}^{12}.$$

Case a. $\alpha, \beta, \gamma, b \neq 0$. By elementary transformations the matrix (7) can be put in the form

$$\text{diagonal}\{b, b, (\alpha-\lambda), -(\alpha-\lambda), \delta, \delta\} \tag{8}$$

where $\delta = b^2+(\gamma-\lambda)(\beta-\lambda)$.

Case a(i). δ is not divisible by $(\alpha-\lambda)$, i.e. $2\alpha+\beta\gamma+b^2 \neq 0$. In this case the matrix (7) is equivalent to

$$\text{diagonal}\{1, 1, 1, 1, (\lambda-\alpha)\delta, (\lambda-\alpha)\delta\}. \tag{9}$$

The invariant factors of the matrix (7) are given by

$$\begin{aligned} &E_1 = E_2 = E_3 = E_4 = 1 \\ &E_5 = E_6 = (\lambda-\alpha)\left(\lambda - \frac{\gamma+\beta+\{(\gamma-\beta)^2-4b^2\}^{1/2}}{2}\right)\left(\lambda - \frac{\gamma+\beta-\{(\gamma-\beta)^2-4b^2\}^{1/2}}{2}\right) \end{aligned} \tag{10}$$

so that the Segre characteristic of the matrix (7) is [(1, 1)(1, 1)(1, 1)], which implies that the metric (1) is of Petrov (1954) type I. When $(\gamma-\beta)^2-4b^2 = 0$, the invariant factors are

$$\begin{aligned} E_1 &= E_2 = E_3 = E_4 = 1 \\ E_5 &= E_6 = (\lambda-\alpha)(\lambda+\alpha)^2 \end{aligned} \tag{11}$$

and the Segre characteristic is [(1, 1)(2, 2)]. Hence the metric (1) is of type II in this case.

Case a(ii). δ is divisible by $(\lambda-\alpha)$. In this case the matrix (7) is equivalent to

$$\text{diagonal}\,\{1, 1, (\lambda-\alpha), (\lambda-\alpha), (\lambda+2\alpha)(\lambda-\alpha), (\lambda+2\alpha)(\lambda-\alpha)\}. \tag{12}$$

The invariant factors are

$$\begin{aligned} E_1 &= E_2 = 1, \qquad E_3 = E_4 = (\lambda-\alpha) \\ E_5 &= E_6 = (\lambda-\alpha)(\lambda+2\alpha) \end{aligned} \tag{13}$$

so that the Segre characteristic is [(1, 1, 1, 1)(1, 1)]. Therefore the metric (1) is of type I.

Case b. $b = 0$ and $\alpha, \beta, \gamma \neq 0$. In this case the matrix is given by

$$\text{diagonal}\,\{\alpha-\lambda, \beta-\lambda, \gamma-\lambda, -(\beta-\lambda), -(\alpha-\lambda), -(\gamma-\lambda)\}. \tag{14}$$

Case b(i). $\alpha \neq \beta \neq \gamma$. In this case the matrix (14) is equivalent to

$$\text{diagonal}\,\{1, 1, 1, 1, (\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma), (\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)\}. \tag{15}$$

The invariant factors are given by

$$\begin{aligned} E_1 &= E_2 = E_3 = E_4 = 1 \\ E_5 &= E_6 = (\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma) \end{aligned} \tag{16}$$

so that the Segre characteristic in this case is [(1, 1)(1, 1)(1, 1)] which implies that the metric (1) is of type I.

Case b(ii). Any two of α, β, γ are equal, $\beta = \gamma$ (say). In this case the matrix (14) is equivalent to

$$\text{diagonal}\,\{1, 1, \lambda-\beta, \lambda-\beta, (\lambda-\beta)(\lambda-\alpha), (\lambda-\beta)(\lambda-\alpha)\}. \tag{17}$$

The invariant factors are given by

$$\begin{aligned} E_1 &= E_2 = 1, \qquad E_3 = E_4 = (\lambda-\beta) \\ E_5 &= E_6 = (\lambda-\alpha)(\lambda-\beta). \end{aligned} \tag{18}$$

The Segre characteristic is [(1, 1, 1, 1)(1, 1)]. Hence the metric is of type I in this case.

Case c. $\alpha = 0, \beta, \gamma, b \neq 0$, this implies that $\beta = -\gamma$. In this case the λ matrix is given by

$$\begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta-\lambda & 0 & 0 & 0 & b \\ 0 & 0 & -(\beta+\lambda) & 0 & b & 0 \\ 0 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & b & 0 & -(\beta-\lambda) & 0 \\ 0 & b & 0 & 0 & 0 & \beta+\lambda \end{bmatrix} \simeq \text{diagonal}\,(b, b, -\lambda, \lambda, \psi, \psi) \tag{19}$$

where $\psi = \lambda^2+b^2-\beta^2$.

Case c(i). ψ is not divisible by λ, i.e. $\beta^2-b^2 \neq 0$. The above matrix is equivalent to

$$\text{diagonal}\,(1, 1, 1, 1, \lambda\psi, \lambda\psi). \tag{20}$$

The invariant factors are

$$\begin{aligned} E_1 &= E_2 = E_3 = E_4 = 1 \\ E_5 &= E_6 = \lambda\{\lambda-(\beta^2-b^2)^{1/2}\}\{\lambda+(\beta^2-b^2)^{1/2}\}. \end{aligned} \tag{21}$$

The Segre characteristic is [(1, 1)(1, 1)(1, 1)]. Hence the metric is of type I.

Case c(ii). $b^2-\beta^2 = 0$, i.e. ψ is divisible by λ. In this case the matrix (19) is equivalent to

$$\text{diagonal}\,(1, 1, \lambda, \lambda, \lambda^2, \lambda^2). \tag{22}$$

The invariant factors are

$$E_1 = E_2 = 1, \qquad E_3 = E_4 = \lambda, \qquad E_5 = E_6 = \lambda^2 \tag{23}$$

so that the Segre characteristic is [(1, 1)(2, 2)]. Therefore the metric is of type II.

Case d. $\beta = 0$, $\alpha, \gamma, b \neq 0$, this implies that $\alpha = -\gamma$. In this case the λ matrix is

$$\begin{bmatrix} \alpha-\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 & b \\ 0 & 0 & -(\alpha+\lambda) & 0 & b & 0 \\ 0 & 0 & 0 & -(\alpha-\lambda) & 0 & 0 \\ 0 & 0 & b & 0 & \lambda & 0 \\ 0 & b & 0 & 0 & 0 & \alpha+\lambda \end{bmatrix} \simeq \text{diagonal}\,\{1, 1, 1, 1, \phi(\alpha-\lambda), \phi(\alpha-\lambda)\} \tag{24}$$

where $\phi = \lambda^2+\lambda\alpha+b$. The invariant factors are

$$\begin{aligned} &E_1 = E_2 = E_3 = E_4 = 1 \\ &E_5 = E_6 = (\lambda-\alpha)\left\{\lambda+\frac{\alpha+(\alpha^2-4b)^{1/2}}{2}\right\}\left\{\lambda+\frac{\alpha-(\alpha^2-4b)^{1/2}}{2}\right\}. \end{aligned} \tag{25}$$

The Segre characteristic is [(1, 1)(1, 1)(1, 1)] so that the metric (1) is of type I. When $\alpha^2-4b = 0$, the invariant factors are

$$E_1 = E_2 = E_3 = E_4 = 1, \qquad E_5 = E_6 = (\lambda-\alpha)(\lambda+\tfrac{1}{2}\alpha)^2 \tag{26}$$

and the Segre characteristic is [(1, 1)(2, 2)]. Hence the metric is of type II.

Case e. $\alpha = 0$, $b = 0 \Rightarrow \beta = -\gamma$. In this case the λ matrix is given by

$$\text{diagonal}\,\{-\lambda, \beta-\lambda, -(\beta+\lambda), \lambda, -(\beta-\lambda), \beta+\lambda\} \tag{27}$$

which is equivalent to

$$\text{diagonal}\,\{1, 1, 1, 1, \lambda(\beta-\lambda)(\beta+\lambda), \lambda(\beta-\lambda)(\beta+\lambda)\}. \tag{28}$$

Hence the Segre characteristic is [(1, 1)(1, 1)(1, 1)] so that the metric is of type I.

Case f. $\alpha = \beta = 0$, $b \neq 0 \Rightarrow \gamma = 0$. In this case the λ matrix is given by

$$\begin{bmatrix} -\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 & b \\ 0 & 0 & -\lambda & 0 & b & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & b & 0 & \lambda & 0 \\ 0 & b & 0 & 0 & 0 & \lambda \end{bmatrix} \simeq \text{diagonal}\,\{1, 1, 1, 1, \lambda(\lambda^2+b^2), \lambda(\lambda^2+b^2)\}. \tag{29}$$

The invariant factors are

$$\begin{aligned} &E_1 = E_2 = E_3 = E_4 = 1 \\ &E_5 = E_6 = \lambda(bi+\lambda)(-bi+\lambda). \end{aligned} \tag{30}$$

The Segre characteristic is [(1, 1)(1, 1)(1, 1)]. Hence the metric is of type I.

The above six cases exhaust the non-trivial possibilities. If α, β, γ and b all vanish, $C_{hijk} \equiv 0$ and the metric becomes conformally flat.

3. Perfect fluid considerations

Here we consider the metric potentials to be functions of t alone. In this case the non-zero components of the energy–momentum tensor of the metric (1) are as follows:

$$\begin{aligned}
-8\pi T_1{}^1 &= -\frac{1}{A^2}\left(\frac{B_{44}}{B}+\frac{C_{44}}{C}\right)+\frac{1}{A^2}\left(\frac{A_4B_4}{AB}+\frac{A_4C_4}{AC}-\frac{B_4C_4}{BC}\right)\\
-8\pi T_2{}^2 &= -\frac{1}{A^2}\left(\frac{A_{44}}{A}+\frac{C_{44}}{C}\right)+\frac{A_4{}^2}{A^4}\\
-8\pi T_3{}^3 &= -\frac{1}{A^2}\left(\frac{A_{44}}{A}+\frac{B_{44}}{B}\right)+\frac{A_4{}^2}{A^4}\\
-8\pi T_4{}^4 &= -\frac{1}{A^2}\left(\frac{A_4B_4}{AB}+\frac{A_4C_4}{AC}+\frac{B_4C_4}{BC}\right).
\end{aligned} \tag{31}$$

In order that the metric may admit perfect fluid distribution we must have

$$T_1{}^1 = T_2{}^2 = T_3{}^3. \tag{32}$$

Equation (32) gives

$$\frac{B_{44}}{B} = \frac{C_{44}}{C} \tag{33}$$

and

$$\frac{A_{44}}{A}-\frac{A_4{}^2}{A^2}+\frac{A_4}{A}\left(\frac{B_4}{B}+\frac{C_4}{C}\right) = \frac{B_{44}}{B}+\frac{B_4C_4}{BC}. \tag{34}$$

We have

$$\left(\frac{B}{C}\right)_{44} = \frac{CB_{44}-BC_{44}}{C^2}-2\frac{C_4}{C}\left(\frac{B}{C}\right)_4. \tag{35}$$

Equations (34) and (35) give

$$\frac{C}{B}\left(\frac{B}{C}\right)_{44}+2\frac{C_4}{B}\left(\frac{B}{C}\right)_4 = \frac{B_{44}}{B}-\frac{C_{44}}{C} = 0. \tag{36}$$

By putting $B = \mu c$, we get

$$\frac{\mu_{44}}{\mu_4}+2\frac{C_4}{C} = 0. \tag{37}$$

Hence

$$\left(\frac{B}{C}\right)_4 = \frac{K}{C^2}$$

i.e.

$$B = C\int\frac{K}{C^2}\,\mathrm{d}t+LC \tag{38}$$

where K and L are arbitrary constants. From equation (34) we obtain

$$\left(\frac{A_4}{A}\right)_4+\frac{A_4}{A}\left(\frac{B_4}{B}+\frac{C_4}{C}\right) = \frac{B_{44}}{B}+\frac{B_4C_4}{BC}$$

or

$$\frac{A_4}{A} = \frac{1}{C^2}\exp(-K\textstyle\int \mathrm{d}t/BC)\displaystyle\int C^2\left(\frac{B_{44}}{B}+\frac{C_4B_4}{BC}\right)\exp(K\textstyle\int \mathrm{d}t/BC)\,\mathrm{d}t+\displaystyle\frac{M}{C^2}\exp(-K\textstyle\int \mathrm{d}t/BC). \tag{39}$$

Hence

$$A = N\exp(\textstyle\int Y\,\mathrm{d}t) \tag{40}$$

where

$$Y = \frac{1}{C^2}\exp(-K\textstyle\int \mathrm{d}t/BC)\displaystyle\int C^2\left(\frac{B_{44}}{B}+\frac{B_4C_4}{BC}\right)\exp(K\textstyle\int \mathrm{d}t/BC)\,\mathrm{d}t+\frac{M}{C^2}\exp(-K\textstyle\int \mathrm{d}t/BC) \tag{41}$$

and M, N are constants of integration. Therefore, the metric (1) reduces to

$$\mathrm{d}s^2 = \{N\exp(\textstyle\int Y\,\mathrm{d}t)\}^2(\mathrm{d}t^2-\mathrm{d}x^2)-\left(C\displaystyle\int\frac{K}{C^2}\,\mathrm{d}t+LC\right)^2\mathrm{d}y^2-C^2\,\mathrm{d}z^2. \tag{42}$$

The components of $T_j^{\ i}$ for (42) are given by

$$\begin{aligned} -8\pi T_1^{\ 1} = -8\pi T_2^{\ 2} = -8\pi T_3^{\ 3} &= \frac{1}{A^2}\left\{\frac{C_{44}}{C}-\left(\frac{A_4}{A}\right)_4\right\} \\ 8\pi T_4^{\ 4} &= \frac{1}{A^2}\left\{\frac{C_{44}}{C}+\frac{2B_4C_4}{BC}-\left(\frac{A_4}{A}\right)_4\right\}. \end{aligned} \tag{43}$$

For the perfect fluid distribution we have

$$8\pi T_j^{\ i} = (\rho+p)v^iv_j-pg_j^{\ i} \tag{44}$$

where $v^iv_i = 1$. Equation (44) leads to

$$\begin{gathered} v^1 = v^2 = v^3 = 0 \\ v^4 = \frac{1}{N\exp(\int Y\,\mathrm{d}t)}, \qquad p = \frac{1}{A^2}\left\{\frac{C_{44}}{C}-\left(\frac{A_4}{A}\right)_4\right\} \end{gathered} \tag{45}$$

and

$$\rho = \frac{1}{A^2}\left\{\frac{2B_4C_4}{BC}+\frac{C_{44}}{C}-\left(\frac{A_4}{A}\right)_4\right\}.$$

In order that density and pressure are positive and $\rho \geqslant 3p$, we must have

$$\frac{C_{44}}{C} > \left(\frac{A_4}{A}\right)_4$$

and

$$\frac{B_4C_4}{BC}+\frac{A_4}{A} \geqslant \frac{C_{44}}{C}. \tag{46}$$

For the case of an incoherent matter field, i.e. when the pressure is zero, the problem has already been solved by Heckmann and Schucking (1962). Another special case of the above perfect fluid distribution has already been reported by the authors (Singh and Singh 1968) as a plane symmetric cosmological model.

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